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Validation of the Variational Asymptotic Beam Sectional Analysis

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The computer program Variational Asymptotic Beam Section Analysis (VABS) uses the variational asymptotic method to split a three-dimensional nonlinear elasticity problem into a two-dimensional linear cross-sectional analysis and a one-dimensional, nonlinear beam problem. This is accomplished by taking advantage of certain small parameters inherent to beam-like structures. VABS is able to calculate the one-dimensional cross-sectional stiffness constants, with transverse shear and Vlasov refinements, for initially twisted and curved beams with arbitrary geometry and material properties. Several validation cases are presented. First, an elliptic bar is modeled with transverse shear refinement using the variational asymptotic method, and the solution is shown to be identical to that obtained from the theory of elasticity. The shear center locations calculated by VABS for various cross sections agree well with those obtained from common engineering analyses. Comparisons with other composite beam theories prove that it is unnecessary to introduce ad hoc kinematic assumptions to build an accurate beam model. For numerical validation values of the one-dimensional variables are extracted from an ABAQUS model and compared with results from a one-dimensional beam analysis using cross-sectional constants from VABS. Furthermore, point-wise three-dimensional stress and strain fields are recovered using VABS, and the correlation with the three-dimensional results from ABAQUS is excellent. Finally, classical theory is shown to be insufficient for general-purpose beam modeling. Appropriate refined theories are recommended for some classes of problems.

Introduction

IN the past 20 years composite materials have found increasing applications in aerospace engineering. Many aircraft components, such as rotor blades and high-aspect-ratio wings, can be modeled as one-dimensional beam problems, thus leading to much simpler governing equations and easy interpretation of the results. To take advantage of this geometric feature without loss of accuracy, one has to capture the behavior associated with the eliminated two dimensions (the cross-sectional coordinates). Especially in composite beams, there can be elastic couplings among all forms of global deformation and among both in- and out-of-plane components of the cross-sectional warping displacement. This generally results in a fully populated matrix of cross-sectional stiffness properties.

The ability to analyze and design composite beam structures accurately lags well behind the ability to manufacture them. The only accurate and reliable analysis tools currently availably to industry are costly, labor-intensive, three-dimensional finite element analyses (FEA). Although less costly beam modeling tools are available, they usually cannot correctly model beams with arbitrary cross-sectional materials and geometries. For example, conventional beam analyses often fail to accurately calculate the uncoupled torsional stiffnesses, and they cannot predict important elastic couplings in composite structures that can be exploited by designers to improve aeroelastic behavior. However, one method for which preliminary results have shown to be quite accurate is the Variational Asymptotic Beam Section Analysis (VABS), originally developed by Cesnik and Hodges[1] and Yu et al.[2] VABS holds great promise for meeting industry's requirements for an efficient, reliable analysis tool for composite beams. This paper presents results from studies that seek to validate VABS. These studies are essential in demonstrating that VABS indeed has accuracy and analysis flexibility comparable to those of the more costly, general-purpose three-dimensional FEA tools such as ABAQUS.

The variational asymptotic method (VAM)[3] is the mathematical basis of VABS and is used to split a general three-dimensional nonlinear elasticity problem for a beamlike structure into a two-dimensional linear cross-sectional analysis and a one-dimensional nonlinear beam analysis. This method exhibits the merits of both variational (systematic) and asymptotic (without ad hoc kinematic assumptions) methods. It allows one to replace a three-dimensional structural model with a reduced-order model in terms of an asymptotic series of certain small parameters inherent to the structure. For the present problem these parameters include the maximum strain in the beam, the ratio of the characteristic diameter of the cross section to the wavelength of the axial deformation (h/ℓ), and the ratio of the characteristic diameter of the cross section to the magnitude of the radius of initial curvature/twist (h/R). VAM applies the asymptotic expansion to the energy functional instead of the system of differential equations.[4] Hence, dropping a small term in the functional is equivalent to neglecting such quantities in several differential equations simultaneously. This implies that, when applicable, VAM is more compact and less cumbersome than standard asymptotic methods.

VABS was first mentioned in Ref. 5. Its development over the past 10 years is described in Refs. 2 and 6–10. VABS can perform a classical analysis for beams with initial twist and curvature with arbitrary reference cross sections. VABS is also capable of capturing the trapeze and Vlasov effects, which are useful for specific beam applications. VABS is now able to calculate the one-dimensional stiffness matrix with transverse shear refinement for any initially twisted and curved, inhomogeneous, anisotropic beam with arbitrary geometry and material properties. Finally, VABS can recover the three-dimensional stress and strain fields, if required, such as finding stress concentrations, interlaminar stresses, etc. VABS is a two-dimensional FEA with a typical element library (triangular elements with 3–6 nodes and quadrilateral elements with 4–9 nodes). It is fully modularized and can be easily integrated into any CAD/CAM software. VABS input is highly compatible with formats used in commercial FEA packages, so that any two-dimensional meshed model of a cross section constructed in PATRAN or ANSYS can be converted into an input for VABS with very little extra effort.

Analytical Modeling of an Elliptic Bar via VAM

Because the purpose of this paper is to present results from the validation of VABS, we will not present the details of the formulation

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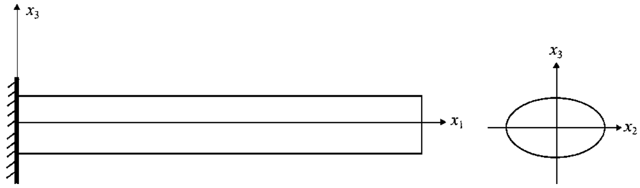

Fig. 1 Sketch of a clamped prism.

here. Details for the most general and up-to-date version of the formulation can be found in Ref. 2. To proceed with the validation, we first undertake the modeling of an elliptic prism analytically using VAM and then compare it with the exact elasticity solution. This exercise will give some insight into the VABS theoretical foundation and serve as a benchmark test for numerical validation. Here and throughout the paper Greek indices assume values 2 and 3, and Latin indices assume 1, 2, and 3. Repeated indices are summed over their range except where explicitly indicated. From the general formulas in Ref. 2, one can write the linear three-dimensional strain field Γ_{ij} for a prism (Fig. 1) as

$$\Gamma_{11} = \gamma_{11} + x_3\kappa_2 - x_2\kappa_3 + w_1', \qquad \Gamma_{22} = w_{2,2}$$
$$\Gamma_{33} = w_{3,3}, \qquad 2\Gamma_{12} = w_{1,2} - x_3\kappa_1 + w_2'$$
$$2\Gamma_{13} = w_{1,3} + x_2\kappa_1 + w_3', \qquad 2\Gamma_{23} = w_{3,2} + w_{2,3} \tag{1}$$

where w_i are the warping components; γ_{11}, κ_1, and κ_α are one-dimensional generalized strains, which consist of the beam extension, torsion, and bending curvatures, respectively; $(\,)'$ denotes a partial derivative with respect to the beam axial coordinate x_1; and $(\,)_{,\alpha}$ denotes a partial derivative with respect to x_α, the local Cartesian coordinates of the undeformed beam cross-sectional plane.

For an isotropic elastic body with Young's modulus E, shear modulus G and Poisson's ratio ν, twice the three-dimensional strain energy per unit length can be written as[11]

$$2U_{3d} = E\langle\Gamma_{11}^2\rangle + 4G\langle\Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{23}^2\rangle + \frac{E}{(1+\nu)(1-2\nu)}$$
$$\left\langle \begin{Bmatrix} \nu\Gamma_{11} + \Gamma_{22} \\ \nu\Gamma_{11} + \Gamma_{33} \end{Bmatrix}^T \begin{bmatrix} 1-\nu & \nu \\ \nu & 1-\nu \end{bmatrix} \begin{Bmatrix} \nu\Gamma_{11} + \Gamma_{22} \\ \nu\Gamma_{11} + \Gamma_{33} \end{Bmatrix} \right\rangle \tag{2}$$

where the notation

$$\langle \cdot \rangle = \int_S \cdot \, \mathrm{d}x_2 \, \mathrm{d}x_3$$

is used throughout this paper, with S being the cross-sectional plane.

The displacement field is four times redundant as a result of the introduction of the warping displacement. To eliminate this redundancy, four appropriate constraints have to be imposed on the warping field. The most convenient way to introduce such constraints is to eliminate from the warping any contribution to the rigid-body motion of a cross section that is caused by the classical one-dimensional deformation measures (that is, the stretching, torsion, and bending in two directions). This is accomplished in the VABS formulation by choosing

$$\langle w_i \rangle = 0 \tag{3}$$

$$\langle x_2 w_3 - x_3 w_2 \rangle = 0 \tag{4}$$

Such a choice of constraints implies that the one-dimensional variables associated with these deformations do not have to be redefined. Next, the functional Eq. (2) is minimized, subject to the constraints in Eqs. (3) and (4).

The response of an elastic body under a given set of loads is completely determined by its strain energy. Hence, to derive an accurate beam theory one has to construct a one-dimensional model that comes as close as possible to reproducing the three-dimensional strain energy. This dimensional reduction cannot be done exactly. VAM is used to find a one-dimensional energy that would approximate the three-dimensional energy as closely as possible taking advantage of the small parameters. We assume $\varepsilon = \max(|\gamma_{11}|, h|\kappa|)$, and usually it is true that $\varepsilon \ll h/\ell$ and $\varepsilon \ll h/R$. Because h/ℓ and h/R both involve h, we can regard h as the small parameter and expand all unknown functions in asymptotic series of h as

$$w_i = w_i^{(0)} + h w_i^{(1)} + \mathcal{O}(h^2) \tag{5}$$

Considering only terms of order $\mathcal{O}(\varepsilon^2)$ in the strain energy, one can write the dominant terms of the energy expression, including the unknown warping functions $w_i^{(0)}$, as $2U_0$. The variational asymptotic procedure requires one to find the warping functions that minimize this expression. For an elliptic cross section with semi-axes a and b in the directions of x_2 and x_3 respectively, it can be shown that the following warping functions uniquely satisfy all of the constraints and minimize $2U_0$:

$$w_1^{(0)} = \frac{(b^2 - a^2)x_2x_3\kappa_1}{a^2 + b^2} \tag{6}$$

$$w_2^{(0)} = \frac{\nu\kappa_3}{8}\left(b^2 - a^2 + 4x_2^2 - 4x_3^2\right) - \nu x_2(\gamma_{11} + x_3\kappa_2) \tag{7}$$

$$w_3^{(0)} = \frac{\nu\kappa_2}{8}\left(b^2 - a^2 + 4x_2^2 - 4x_3^2\right) + \nu x_3(x_2\kappa_3 - \gamma_{11}) \tag{8}$$

The resulting value of $2U_0$ is the first approximation of the strain energy and coincides with the result of the classical beam theory:

$$2U_0 = \pi abE\gamma_{11}^2 + \frac{\pi a^3b^3G}{a^2+b^2}\kappa_1^2 + \frac{\pi}{4}ab^3E\kappa_2^2 + \frac{\pi}{4}a^3bE\kappa_3^2$$
$$= EA\gamma_{11}^2 + GJ\kappa_1^2 + EI_2\kappa_2^2 + EI_3\kappa_3^2 \tag{9}$$

The resulting result is obtained without any ad hoc assumptions such as "the cross section is rigid in its own plane" or "$\nu = 0$."

To obtain a more refined theory, one can repeat the same procedure in order to find the dominant terms of the strain energy up to any desired order. For a prismatic beam it is easy to prove that the correction to the strain energy of order h is zero. Denoting the dominant terms of the energy in the order of h^2 as $2U_1$, we find that they contain the derivatives of the unknown functions with respect to the axial coordinate x_1. Integration by parts with respect to x_1 of $2U_1$ does not affect the final result, but it does allow the strain energy to be expressed in terms of the unknown functions instead of their derivatives. Carrying out a minimization procedure, which is similar to the one conducted to obtain the "classical" theory, one obtains the warping field that satisfies all of the constraints and minimizes the second-order strain energy. The warpings $w_2^{(1)}$ and $w_3^{(1)}$ are identically zero, whereas $w_1^{(1)}$ is a complicated third-order polynomial, not given here for the sake of brevity. The minimized value of $2U_1$ is

$$\frac{2U_1}{EAa^4} = \frac{(3\rho^2 + \rho^4)[\nu^2 + 2\rho^2(1+\nu)^2 + 5\rho^4(1+\nu)^2]\kappa_2'^2}{12(3+\rho^2)(1+3\rho^2)(1+\nu)}$$
$$+ \frac{(1+3\rho^2)[\rho^4\nu^2 + 2\rho^2(1+\nu)^2 + 5(1+\nu)^2]\kappa_3'^2}{12(3+\rho^2)(1+3\rho^2)(1+\nu)} \tag{10}$$

where $\rho = a/b$ is the aspect ratio.

Although Eq. (10) is an asymptotically correct expression up to the order of h^2, this form of the strain energy would be impractical for use in engineering analysis. The expression involves derivatives of the one-dimensional generalized strain measures with respect to x_1, and applying appropriate boundary conditions would be very difficult.[12]

A Timoshenko-like beam model is free from such drawbacks and can be expressed symbolically as

$$2U = \epsilon^T X \epsilon + 2\epsilon^T F\gamma + \gamma^T G\gamma \tag{11}$$

where $\epsilon = [\gamma_{11} \;\; \kappa_1 \;\; \kappa_2 \;\; \kappa_3]^T$ represents the one-dimensional generalized strains associated with extension, torsion, and bending; and $\gamma = [2\gamma_{12} \;\; 2\gamma_{13}]^T$ is the column matrix of transverse shear strain measures. (It should be noted that ϵ is defined in terms of the cross-sectional rotation variables of the Timoshenko beam model.) Following the general formulation in Ref. 2, one can transform the strain energy that is asymptotically correct up to the order of h^2 into a Timoshenko-like model. Superposing the strain energies, Eqs. (9) and (10), we can rewrite the strain energy in a matrix form as

$$2U = \varepsilon^T A\varepsilon + \varepsilon'^T C\varepsilon' \tag{12}$$

where

$$A = \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & GJ & 0 & 0 \\ 0 & 0 & EI_2 & 0 \\ 0 & 0 & 0 & EI_3 \end{bmatrix} \tag{13}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 \\ 0 & 0 & 0 & C_3 \end{bmatrix} \tag{14}$$

C_2 and C_3 are the coefficients of $\kappa_2'^2$ and $\kappa_3'^2$, respectively, in Eq. (10).

Using the formulas given in Ref. 2, one can finally obtain the Timoshenko model for the present problem as

$$2U = \begin{Bmatrix} \gamma_{11} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}^T \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & GJ & 0 & 0 \\ 0 & 0 & EI_2 & 0 \\ 0 & 0 & 0 & EI_3 \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} + \begin{Bmatrix} 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix}^T \begin{bmatrix} S_2 & 0 \\ 0 & S_3 \end{bmatrix} \begin{Bmatrix} 2\gamma_{12} \\ 2\gamma_{13} \end{Bmatrix} \tag{15}$$

where

$$S_2 = \frac{3a^2(3a^2+b^2)(1+\nu)^2 GA}{2[b^4\nu^2 + 5a^4(1+\nu)^2 + 2a^2b^2(1+\nu)^2]} \tag{16}$$

$$S_3 = \frac{3b^2(a^2+3b^2)(1+\nu)^2 GA}{2[a^4\nu^2 + 5b^4(1+\nu)^2 + 2a^2b^2(1+\nu)^2]} \tag{17}$$

The shear stiffnesses in Eqs. (16) and (17) are identically the same as those obtained from the elasticity solution for a clamped, terminally loaded, elliptical prism.[13] It is obvious that those shear correction factors are the functions of the aspect ratio and Poisson's ratio. Poisson's ratio will affect the shear correction factor significantly, especially when the aspect ratio is small.

One can find the shear stiffnesses for other cross-sectional shapes using the same procedure. However, a closed-form solution for the warping will not in general exist. One can still use the Ritz method to solve the variational problem approximately using any symbolic computing software. The procedure used in VABS is essentially the same, except that we solve the variational problem using finite elements; the convergence to the exact solution mainly depends on the mesh refinement.

Locating Shear Center by VABS

The shear center for a beam is often defined as the point through which a transverse force will only cause bending and transverse shear deformation without twist.[14] The locus of shear centers for all cross sections along the beam axis is called the elastic axis of the beam. It is known that the elastic axis is a natural reference line for describing the elastic deformation of isotropic beams because it leads to simpler resulting governing equations. For anisotropic beams the simplication is less advantageous for reasons explained in the following.

To locate the elastic axis, one has to find the shear center as accurately as possible, especially for thin-walled beams with open cross sections. There the torsional stiffness is much smaller than the bending stiffness, and even a very small error in locating the shear center can result in undesirable twisting, with respect to which the design has to be made robust. For thin-walled beams it is not difficult to locate the shear center for homogeneous beams made from isotropic materials. But for thick-walled or composite beams, it is not possible in general to find a closed-form solution for such a point.

However, if one has obtained an accurate 6×6 stiffness matrix, finding the shear center becomes trivial. Suppose one finds the following beam constitutive law:

$$\begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} \tag{18}$$

To obtain the shear center location, it is sufficient to assume that there are two transverse forces at the tip of a cantilever beam $\hat{F}_2$, $\hat{F}_3$. Hence, $M_1 = \hat{F}_3 e_2 - \hat{F}_2 e_3$, $M_2 = -\hat{F}_3(L-x)$, $M_3 = \hat{F}_2(L-x)$. According to the definition of the shear center, we need to find e_2 and e_3 to locate a position where an application of the transverse forces results in zero twist, that is, $\kappa_1 = 0$. This can be written in terms of loading and stiffness as

$$[S_{24} - S_{44}e_3 + S_{46}(L-x)]\hat{F}_2 + [S_{34} + S_{44}e_2 - S_{45}(L-x)]\hat{F}_3 = 0 \tag{19}$$

Because this equation is valid for any arbitrary $\hat{F}_2$ and $\hat{F}_3$, the location of shear center can be easily obtained:

$$e_2 = -S_{34}/S_{44} + (S_{45}/S_{44})(L-x) \tag{20}$$

$$e_3 = S_{24}/S_{44} + (S_{46}/S_{44})(L-x) \tag{21}$$

This demonstrates that the position of the shear center varies linearly with respect to the axial coordinate and is thus not a cross-sectional property for beams with bending-twist coupling. Clearly, the use of the locus of shear centers as the beam reference line is not feasible in general. However, if there is no bending-twist coupling then the shear center becomes a cross-sectional property; and the use of the locus of the shear centers provides a reference line that decouples bending and twist, hence making it a popular choice. For beams with bending-twist coupling such as composite beams, one can modify the definition of shear center by considering only the twist caused by the shear forces and excluding the twist produced by bending moment through the bending-twist coupling. In such cases the second term in Eqs. (21) and (20) will drop out, and the shear center, by this modified definition, becomes a cross-sectional property. A similar conclusion is reached in Refs. 15 and 16.

As the first example, we investigate the shear center of a channel section (Fig. 2). Using the thin-walled assumption, the elasticity solution for the shear center turns out to be

$$e/b = \tfrac{1}{3} - \tfrac{1}{12}(t/b)^2 + 1\Big/\left[\tfrac{5}{3}(t/b)^2 + \tfrac{7}{3}\right] \tag{22}$$

The calculated location of the shear center, normalized by the thin-walled result, vs the aspect ratio b/t is shown in Fig. 3. It is clear that when the aspect ratio is very large the shear center location calculated from the 6×6 stiffness matrix of VABS converges to the result of thin-walled theory.

There is no known exact solution for the shear center of a solid, homogeneous, isotropic, triangular cross section such as depicted in Fig. 4. The shear center location calculated from VABS vs the aspect ratio b/a is plotted in Fig. 5. When b/a is very small, the triangular section acts as a thin, rectangular section, and the shear center approaches the midpoint. For an equilateral triangular section the shear center is at the centroid. When b/a is very large, the shear center moves toward the vertical edge.

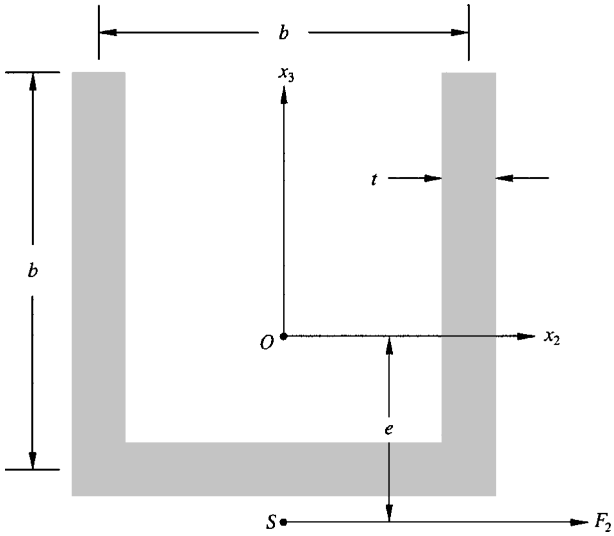

Fig. 2 Sketch of a channel section.

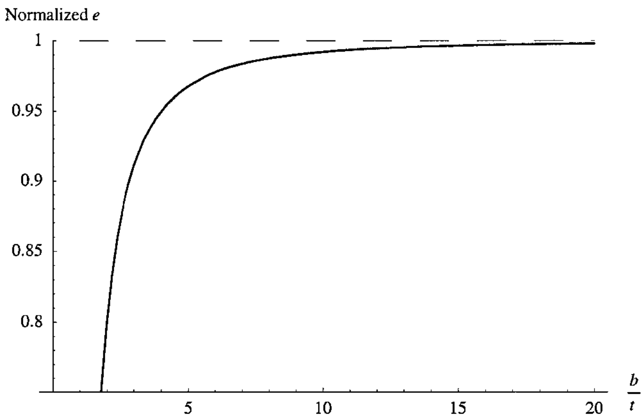

Fig. 3 Shear center location of channel section.

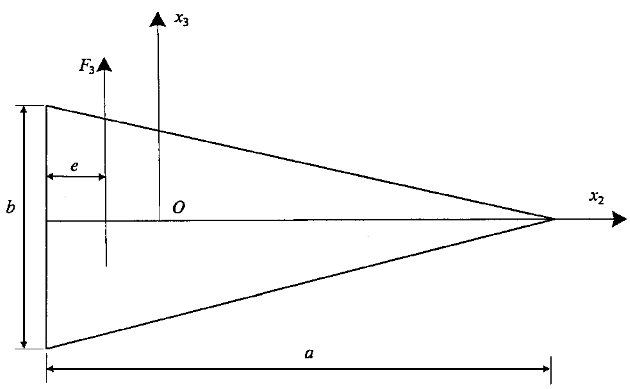

Fig. 4 Sketch of triangular section.

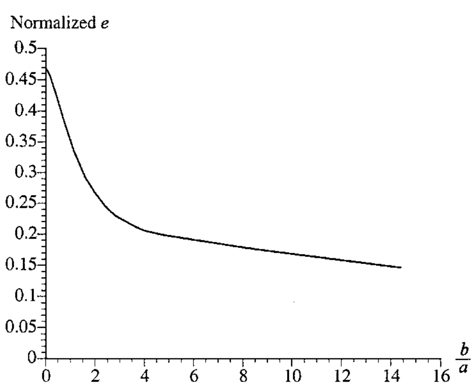

Fig. 5 Shear center location of triangular section.

VABS vs Other Theories and Three-Dimensional FEA

An extensive comparison with other composite beam theories was presented in Ref. 17. The validity of various beam-modeling schemes and the relative importance of various assumptions on which they are based were discussed therein. It was shown that an asymptotically correct model, which alone represents the three-dimensional model as closely as possible, can serve as a standard to evaluate other models. Here only the popular uniaxial stress assumption is revisited. This assumption yields a much simpler model as a result of the elimination of the in-plane warping from the formulation, and it would appear to be quite reasonable to believe that the beam constants based on this assumption would differ only slightly from their asymptotically correct counterparts. However, there are some cases where the difference is not negligible at all. Indeed, it can be verified that the hypothesis of vanishing stresses in the plane of cross section only holds for isotropic beams.

To see the consequences of making this incorrect assumption, consider a box beam as shown in Fig. 6, with s as the positive fiber direction and n as the normal direction indicating the stacking sequence. The geometry and material properties are as listed in Table 1 with l as the fiber direction, t the transverse direction, and n the direction normal to the ply plane. Figure 7 shows that the torsional rigidity is severely underpredicted when this assumption is invoked.

The analytical solution is based on the thin-walled solution starting from shell theory. It yields the same result as VABS because both of them are based on the variational asymptotic method. There is no unique beam theory for a given order of asymptotical correctness. The expansions of two asymptotically correct theories differ in their higher-order terms, the ones that are not asymptotically correct. However, one must be consistent with the order of small parameters and extremely careful in the neglect of small quantities when carrying out this approach analytically. The upper curve, which overpredicts the torsional rigidity, is also from a variational asymptotic approach in which κ_{ss} was neglected by mistake.

Comparing only the values of beam stiffness matrix can be very misleading if two models are derived from different methodologies. One such example is Table 9 in Ref. 9. A correct basis for comparison is to find how accurately two different one-dimensional models approximate the three-dimensional strain energy. There are

Table 1 Properties of thin-walled box beam

Property	Value
Outer dimensions	
Width	$a = 0.953$ in.
Height	$b = 0.53$ in.
Thickness	$h = 0.03$
Right and upper wall	$(\theta)_3/(-\theta)_3$
Left and lower wall	$(-\theta)_3/(\theta)_3$
Material properties	$E_l = 20.59 \times 10^6$ psi
	$E_t = 1.42 \times 10^6$ psi
	$G_{lt} = 8.7 \times 10^5$ psi
	$G_{tn} = 6.96 \times 10^5$ psi
	$\nu_{lt} = \nu_{tn} = 0.42$

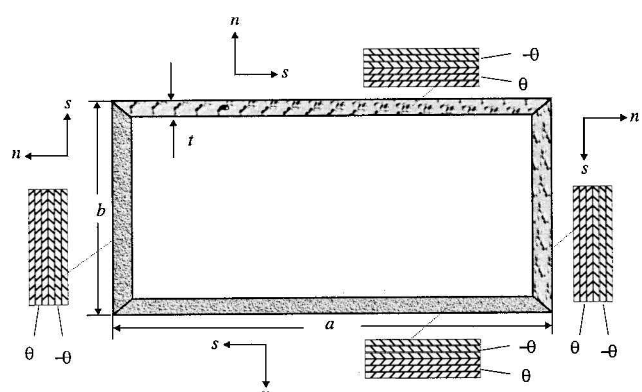

Fig. 6 Sketch of the cross section of a box beam.

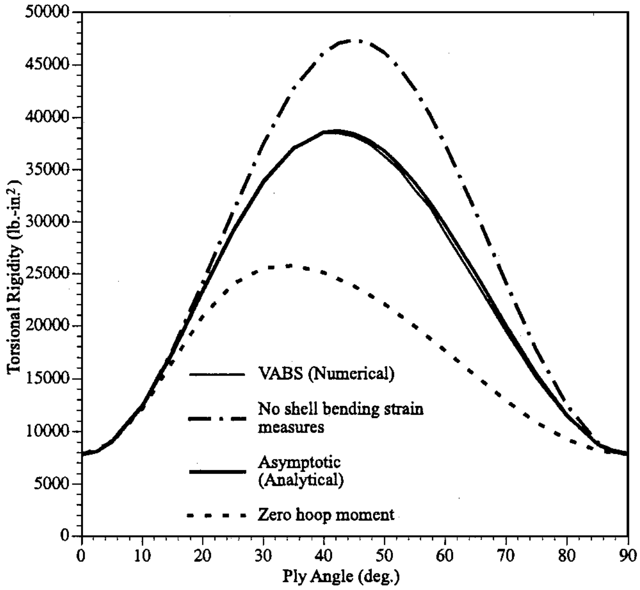

Fig. 7 Torsional rigidity for a box beam.

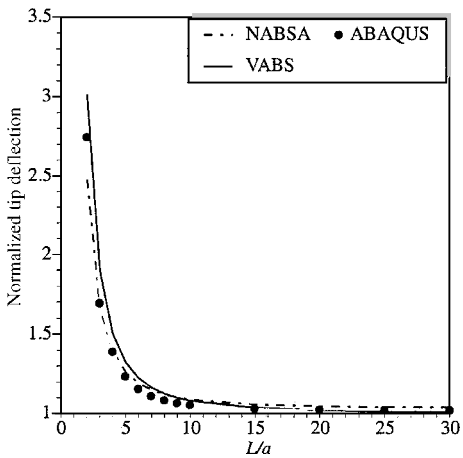

Fig. 8 Comparison of NABSA, VABS, and three-dimensional FEA results.

two ways to carry out this comparison. One is to extract the one-dimensional information from three-dimensional results based on the definition of one-dimensional variables of the beam models. For the very different stiffness matrices in Table 9 of Ref. 9, one natural way to assess the difference is to compare a one-dimensional result (like a tip defection) using various stiffness models with a three-dimensional solution (obtained from the authors of Ref. 18). The plots are shown in Fig. 8. The dashed line shows results from the NABSA[19] model, and the solid line shows the results from the VABS model. These curves are very close. The VABS curve is closer to the three-dimensional results when the beam is slender, and the NABSA results do a bit better in the limit of the beam becoming fat (that is, ceasing to be a beam). The symbols depict results from running ABAQUS with degrees of freedom on the order of 10^5 for the longer beams. Both models capture the essential behavior, and there is very little difference between the results predicted by VABS and those from three-dimensional FEA.

Another way to compare these predictions is to recover the three-dimensional stress or strain fields from the one-dimensional models and compare them with the stress or strain from the three-dimensional FEA. Recall that, as a consequence of decoupling the three-dimensional elasticity problem, the three-dimensional strain field can be recovered by one-dimensional strain measures, cross-sectional warping, and their derivatives as follows:

$$\Gamma = (1/h)\Gamma_h w + \Gamma_\varepsilon \varepsilon + \Gamma_R w + \Gamma_l w' \tag{23}$$

where $\Gamma = [\Gamma_{11}\ 2\Gamma_{12}\ 2\Gamma_{13}\ \Gamma_{22}\ 2\Gamma_{23}\ \Gamma_{33}]^T$, $w = [w_1\ w_2\ w_3]^T$, $\varepsilon = [\gamma_{11}\ \kappa_1\ \kappa_2\ \kappa_3\ ]^T$. All of the operators in Eq. (23) can be found in Ref. 2.

The warping is a function of the one-dimensional generalized strain measures and the discretized cross-sectional warping already obtained by VABS. The only remaining step is to solve the beam problem in order to calculate the one-dimensional generalized strain measures for a given set of loads. A geometrically exact, nonlinear one-dimensional theory is a natural and consistent outcome of the dimensional reduction.[20] However, for the purposes of our validation studies here, it is sufficient to use a linear version of the one-dimensional theory taken directly from Ref. 20:

$$F' + \tilde{k}F + f = 0, \qquad M' + \tilde{k}M + \tilde{e}_1 F + m = 0 \tag{24}$$

where the operator ~ is defined as $\tilde{\cdot}_{ij} = -e_{ijl}\cdot_l$; $k = \lfloor k_1\ \ k_2\ \ k_3 \rfloor^T$ with k_1 the initial twist; k_α the components of initial curvature along x_α expressed in the beam cross-sectional frame; $e_1 = \lfloor 1\ \ 0\ \ 0 \rfloor^T$; the column matrices $F(x_1)$ and $M(x_1)$ contain the measure numbers of cross-sectional force and moment stress resultants measured in the beam cross-sectional frame; and the column matrices $f(x_1)$ and $m(x_1)$ contain the applied distributed forces and moments along the beam expressed in the beam cross-sectional frame. Subject to any given boundary and loading conditions and given the one-dimensional stiffness matrix calculated by VABS and the kinematical equations

$$\gamma = u' + \tilde{k}u + \tilde{e}_1\theta, \qquad \kappa = \theta' + \tilde{k}\theta = 0 \tag{25}$$

where $u(x_1)$ is the column matrix of displacement measures expressed in the beam cross-sectional frame and $\theta(x_1)$ is the column matrix of infinitesimal cross-sectional rotations, one can find all of the one-dimensional variables. Making use of the warping field just obtained, from Eq. (23) one can the find the three-dimensional strain field. Finally, the three-dimensional stress field can be recovered by using the general material constitutive law. VABS does not restrict the three-dimensional material properties in any way, so that one can use as many as 21 independent constants.

VABS can recover the three-dimensional stress-strain field for many classical problems solvable by elasticity theory, such as the Saint-Venant solution and flexure problem for elliptical and rectangular prisms. We will now investigate some cases where the exact elasticity solution cannot be obtained. Naturally, a comparison is made with results from three-dimensional FEA.

For classical modeling we take a rectangular cantilever beam with $0 \le x_1 \le 20$, $-0.5 \le x_2 \le 0.5$, and $-1 \le x_3 \le 1$ (dimensions are in inches), under a tip twisting moment of magnitude 1 lb-in. The cross section is divided into four layers made with two different isotropic materials along the x_3 direction. This section is discretized with 192 eight-noded quadrilateral elements. The Young's modulus E for the top and bottom layers is 2.6×10^7 psi, and for the middle two it is 2.6×10^6 psi; Poisson's ratio is 0.3. The recovered three-dimensional stress σ_{12} is plotted in Fig. 9. From the plot one can observe that there is a stress discontinuity along the boundary between the two different materials. To compare the results with three-dimensional FEA using ABAQUS, we plot the three-dimensional strain and stress along the height through the middle of the cross section in Figs. 10 and 11. The solid lines are the results recovered from VABS, and the symbols are from ABAQUS. Although the strain is continuous, there are obvious jumps in the stress distribution at the junction of the two materials, as expected. The VABS results are right on top of those from the three-dimensional FEA.

Another example is the Timoshenko-like modeling of an initially curved ($k_3 = 0.01$ rad/in.), isotropic, rectangular, cantilever beam with the same cross section ($E = 2.6 \times 10^7$ psi, $\nu = 0.3$) subject to a unit tip shear force in the x_2 direction. This time the cross section is meshed with 200 eight-noded quadrilateral elements. The reference line of this beam spans an arc of 10 deg. The three-dimensional stress component σ_{12} at the midspan is plotted in Fig. 12. The agreement with the ABAQUS result is also excellent, as one can observe from Figs. 13 and 14. To achieve the same accuracy, the ABAQUS model uses 15,872 C3D20R three-dimensional brick elements.

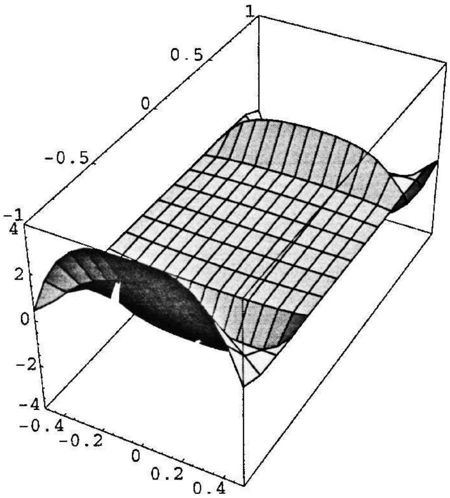

Fig. 9 Three-dimensional stress σ_{12} distribution under torsional loading.

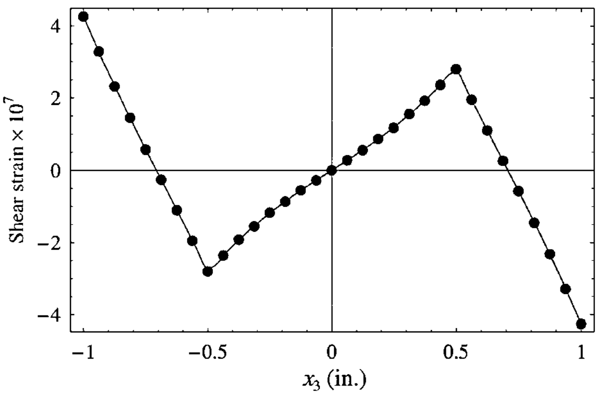

Fig. 10 Three-dimensional strain $2\Gamma_{12}$ along $x_2 = 0$.

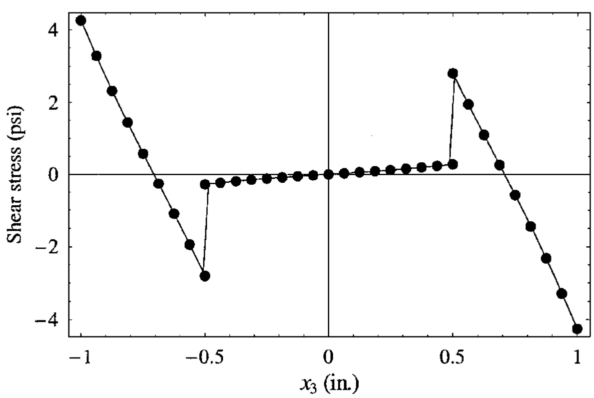

Fig. 11 Three-dimensional stress σ_{12} along $x_2 = 0$.

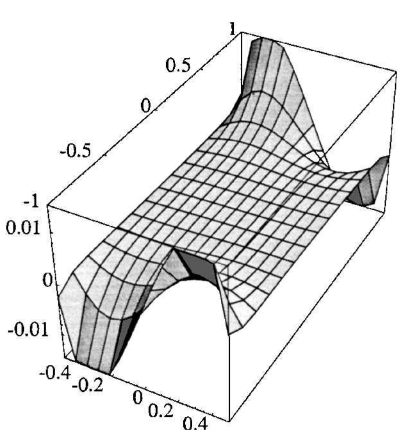

Fig. 12 Three-dimensional stress σ_{12} distribution under shear loading.

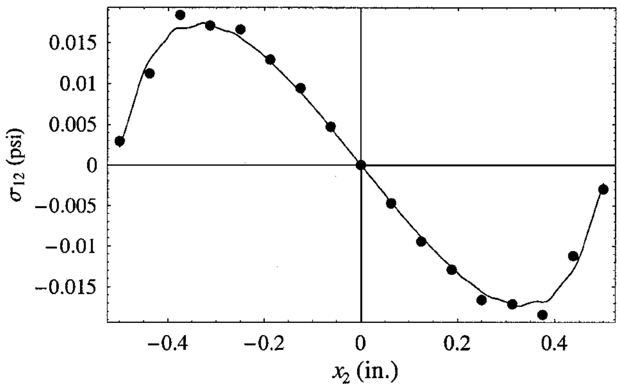

Fig. 13 Three-dimensional stress σ_{12} at $x_3 = 1$.

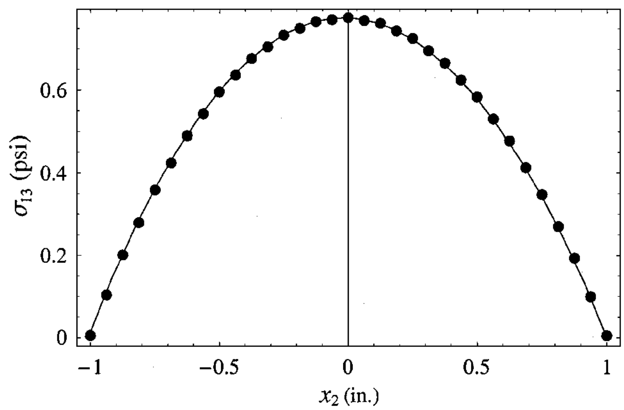

Fig. 14 Three-dimensional stress σ_{13} at $x_2 = 0.5$.

Additional Considerations for Refined Beam Theories

A classical theory that accounts for extension, torsion, and bending in two directions provides excellent results in many situations, namely, when the beam is slender, is not thin-walled with open section, and undergoes long-wavelength deformation (that is, static behavior in response to loads that vary slowly over the length or low-frequency modes of vibration). However, a refined theory is required for high accuracy in other situations. VABS can calculate stiffness and recovery relations for the following two refinements: 1) the Timoshenko (transverse shear) refinement, needed for short-wavelength deformation; and 2) the Vlasov refinement, typically needed for thin-walled, open-section beams when the torsional stiffness is very small compared to other stiffnesses. VABS is also able to capture the nonlinear Trapeze effect caused by moderate local rotations. The significance of the Timoshenko refinement can be observed from Fig. 8. For example, using a classical model for stubby beams with aspect ratio less than 5 will introduce errors of the order of 32%. Even when the aspect ratio is equal to 10, there is still an error around 8%, which might be not acceptable for some applications. A Timoshenko-like model might also be required for accurate recovery of stresses other than σ_{11}.

Vlasov's theory was developed for thin-walled beams with open cross sections. It addresses the problem that the cross sections of such beams cannot warp freely as assumed in the Saint-Venant solutions. There is a strong restraining effect at the ends of the beam, which will result in an significant increase in the effective torsional rigidity. A well-studied case[21] is the composite I-beam as shown in Fig. 15, where $\theta = 15$ deg, $a = 1$ in., $b = 0.5$ in., $h = 0.04$ in., and the length of the beam is 30 in. The material properties are the same as those listed in Table 1, except $G_{tn} = 8.9 \times 10^5$ psi. This configuration exhibits strong bending-twist coupling, and the torsional rigidity is nearly an order of magnitude less than the coupling. For a unit-twisting moment applied at the tip, results for the tip twist angle and bending slope obtained from the classical and Vlasov models are plotted in Fig. 16. It is shown that classical theory overpredicts the tip angles by a factor of two relative to the corrected result. Although the Vlasov term does not yield much of a correction for the

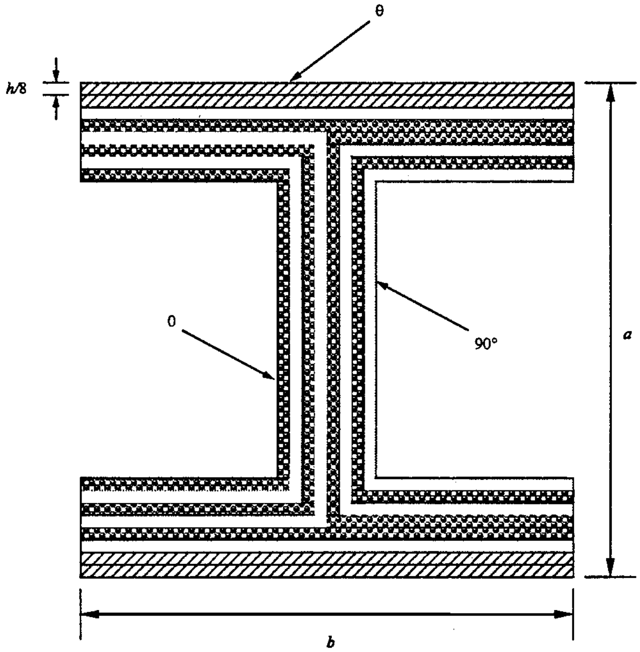

Fig. 15 I-beam configuration and layup.

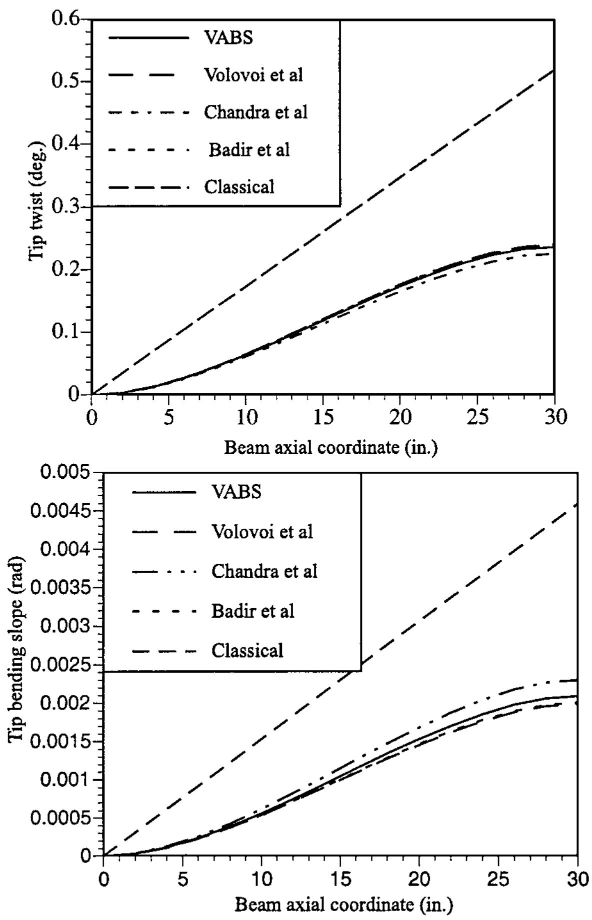

Fig. 16 Tip twist and bending slope under tip torsional load.

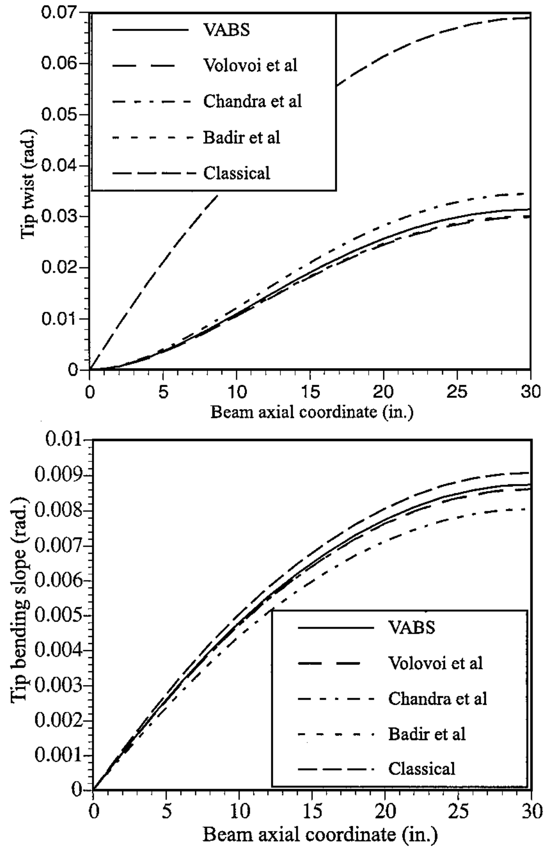

Fig. 17 Tip twist and bending slope under tip shear load.

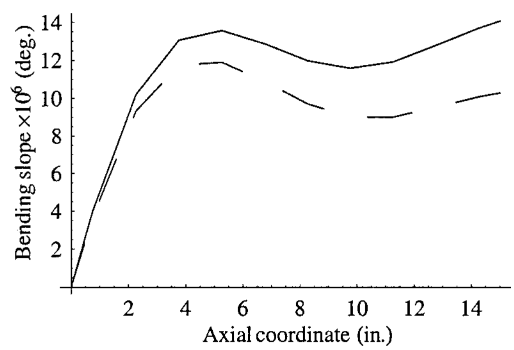

Fig. 18 Bending slope vs axial coordinate (k_1 = 0.2 rad/in.).

tip bending slope for this beam under a unit tip shear force (Fig. 17), there is still a large correction for the tip twist under this loading. There are also other analytical models with the Vlasov refinement plotted in this figure,[21–23] all of which exhibit excellent agreement with the VABS result.

There is still another subtle refinement that has not attracted much attention in the literature. It is shown in Ref. 2 that the initial curvatures and twist will affect the stiffness model and introduce classical and nonclassical couplings. These effects will be more significant for anisotropic beams, which are designed to use the couplings. As pointed out in Ref. 8, analysts sometimes use the stiffness constants from a prismatic beam sectional analysis in the geometrically exact one-dimensional equilibrium and kinematical equations for curved/twisted beams. From the asymptotic point of view, this practice will clearly introduce errors that are sometimes significant. To demonstrate this, a simple isotropic rectangular cross section with dimensions as 2 × 1 in. and Young's modulus as $E = 2.6 \times 10^9$ psi and Poisson's ratio $\nu = 0.3$ is studied. In Fig. 18 results are presented for a beam that is initially twisted and clamped at $x_1 = 0$ and loaded at the tip where $x_1 = L = 15$ in. with $F_3 = 1000$ lb. The dashed line is calculated by using prismatic stiffness model in the geometrically exact one-dimensional governing equations, and the solid line is calculated using the corrected stiffness model. As one can see, there is a large difference (up to 40%) even for a moderate initial twist ($k_1 = 0.2$ rad/in.). This is not surprising because initial twist introduces extension-twist and bending-shear couplings into the stiffness model and also modifies the diagonal terms.

Experience with VABS in a practical setting[24] shows that realistic helicopter rotor blades can be modeled in VABS with a high level of detail (even to the point of actually modeling paint and adhesive layers) using 10,000–30,000 two-dimensional elements. By modern computer standards this is a problem of modest size and makes detailed stress analysis practical and feasible. A comparable level of detail in three-dimensional finite element analysis requires several million elements and about three orders of magnitude more computing time than analyzing the problem using VABS and a suitable beam code.

The present validation is not claimed to be exhaustive. Indeed, given adequate resources, one could carry out a systematic comparison of the three-dimensional FEA and the one-dimensional models (in both of the ways just noted), while carefully maintaining similar mesh geometries and monitoring pointwise behavior of all field variables.

Conclusions

It has been demonstrated herein that VABS can reproduce the results of the theory of elasticity and correctly locate the shear center for beams of arbitrary cross-sectional material and geometry. VABS also can evaluate other theories and maintain an accuracy comparable to that of three-dimensional finite element codes. All of this is very important for preliminary as well as detailed design calculations. VABS shares many features with standard finite element codes. It requires the same kinds of input as found in other finite element codes and takes advantage of standard finite element procedures, including commercially available mesh generation and postprocessing. Moreover, it relies on solution procedures found in standard finite element codes. Although VABS is restricted to beam applications, it provides a level of accuracy comparable to that of standard three-dimensional finite element codes, but with far smaller computing and preprocessing requirements. Moreover, the cross-sectional stiffness matrix, once computed, can be used to solve many beam problems with varying boundary conditions, loading, and motion.

References

[1]Cesnik, C. E. S., and Hodges, D. H., "VABS: A New Concept for Composite Rotor Blade Cross-Sectional Modeling," *Journal of the American Helicopter Society*, Vol. 42, No. 1, 1997, pp. 27–38.

[2]Yu, W., Hodges, D. H., Volovoi, V. V., and Cesnik, C. E. S., "Timoshenko-Like Modeling of Initially Curved and Twisted Composite Beams with Oblique Cross Sections," *International Journal of Solids and Structures* (to be published).

[3]Berdichevsky, V. L., "Variational-Asymptotic Method of Constructing a Theory of Shells," *PMM*, Vol. 43, No. 4, 1979, pp. 664–687.

[4]Le, Khanh Chau, *Vibrations of Shells and Rods*, 1st ed., Springer-Verlag, Berlin, 1999, pp. 45–55.

[5]Hodges, D. H., Atilgan, A. R., Cesnik, C. E. S., and Fulton, M. V., "On a Simplified Strain Energy Function for Geometrically Nonlinear Behaviour of Anisotropic Beams," *Composites Engineering*, Vol. 2, No. 5–7, 1992, pp. 513–526.

[6]Cesnik, C. E. S., and Hodges, D. H., "Stiffness Constants for Initially Twisted and Curved Composite Beams," *Applied Mechanics Reviews*, Vol. 46, No. 11, Pt. 2, 1993, pp. S211–S220.

[7]Cesnik, C. E. S., and Hodges, D. H., "Variational-Asymptotical Analysis of Initially Twisted and Curved Composite Beams," *International Journal for Engineering Analysis and Design*, Vol. 1, No. 2, 1994, pp. 177–187.

[8]Cesnik, C. E. S., and Hodges, D. H., "Stiffness Constants for Composite Beams Including Large Initial Twist and Curvature Effects," *Applied Mechanics Reviews*, Vol. 48, No. 11, Pt. 2, 1995, pp. S61–S67.

[9]Popescu, B., and Hodges, D. H., "On Asymptotically Correct Timoshenko-Like Anisotropic Beam Theory," *International Journal of Solids and Structures*, Vol. 37, No. 3, 1999, pp. 535–558.

[10]Popescu, B., Hodges, D. H., and Cesnik, C. E. S., "Obliqueness Effects in Asymptotic Cross-Sectional Analysis of Composite Beams," *Computers and Structures*, Vol. 76, No. 4, 2000, pp. 533–543.

[11]Hodges, D. H., "Non-Linear Inplane Deformation and Buckling of Rings and High Arches," *International Journal of Non-Linear Mechanics*, Vol. 34, No. 4, 1999, pp. 723–737.

[12]Sutyrin, V. G., "Derivation of Plate Theory Accounting Asymptotically Correct Shear Deformation," *Journal of Applied Mechanics*, Vol. 64, No. 4, 1997, pp. 905–915.

[13]Renton, J. D., "Generalized Beam Theory Applied to Shear Stiffness," *International Journal of Solids and Structures*, Vol. 27, No. 15, 1991, pp. 1955–1967.

[14]Fung, Y. C., *An Introduction to the Theory of Aeroelasticity*, 1st ed., Dover, New York, 1993, pp. 17–19.

[15]Rehfield, L. W., and Atilgan, A. R., "Shear Center and Elastic Axis and Their Usefulness for Composite Thin-Walled Beams," *Proceedings of the American Society for Composites, Fourth Technical Conference*, Blacksburg, VA, 1989, pp. 179–188.

[16]Kosmatka, J. B., "General Behavior and Shear Center Location of Prismatic Anisotropic Beams via Power Series," *International Journal of Solids and Structures*, Vol. 31, No. 3, 1994, pp. 417–439.

[17]Volovoi, V. V., Hodges, D. H., Cesnik, C. E. S., and Popescu, B., "Assessment of Beam Modeling Methods for Rotor Blade Applications," *Mathematical and Computer Modelling*, Vol. 33, No. 10–11, 2001, pp. 1099–1112.

[18]Jung, S. N., Nagaraj, V. T., and Chopra, I., "Refined Structural Model for Thin- and Thick-Walled Composite Rotor Blades," *AIAA Journal*, Vol. 40, No. 1, 2002, pp. 105–116.

[19]Giavotto, V., Borri, M., Mantegazza, P., Ghiringhelli, G., Carmaschi, V., Maffioli, G. C., and Mussi, F., "Anisotropic Beam Theory and Applications," *Computers and Structures*, Vol. 16, No. 1–4, 1983, pp. 403–413.

[20]Hodges, D. H., "A Mixed Variational Formulation Based on Exact Intrinsic Equations for Dynamics of Moving Beams," *International Journal of Solids and Structures*, Vol. 26, No. 11, 1990, pp. 1253–1273.

[21]Chandra, R., and Chopra, I., "Experimental and Theoretical Analysis of Composite I-Beams with Elastic Coupling," *AIAA Journal*, Vol. 29, No. 12, 1991, pp. 2197–2206.

[22]Volovoi, V. V., Hodges, D. H., Berdichevsky, V. L., and Sutyrin, V. G., "Asymptotic Theory for Static Behavior of Elastic Anisotropic I-Beams," *International Journal of Solids and Structures*, Vol. 36, No. 7, 1999, pp. 1017–1043.

[23]Badir, A. M., Berdichevsky, V. L., and Armanios, E. A., "Theory of Composite Thin-Walled Open Cross Section Beams," *Proceedings of the 34th Structures, Structural Dynamics, and Materials Conference*, Pt. 5, AIAA, Reston, VA, 1993, pp. 2761–2770.

[24]Hodges, D. H., Bauchau, O. A., Craig, J. I., and Volovoi, V. V., "Comprehensive Aeromechanical Analysis Using Integrated Analysis Tools," Rotorcraft Industry Technology Association (RITA), Inc., Final Rept. WBS 2000-B-04-17.1.1-A1, Orange, CT, March 2002.

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